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Rogers Dilogarithm in Integrable Systems*

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We discuss some curious aspects of the Rogers dilogarithm appearing in integrable systems in two dimensions.

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1. Introduction

This note is a brief exposition of the appearance of the Rogers dilogarithm function in relation to integrable lattice models and conformal field theory (CFT) in two dimensions. The content is mainly the known facts in refs.[1,2] but also includes a few new informations based on a collaboration with Junji Suzuki.

The Rogers dilogarithm is a function of a variable x defined by

$$L(x) = -\frac{1}{2}\int_0^x \Big(\frac{log(1-y)}{y} + \frac{logy}{1-y}\Big)dy \quad (0 \le x \le 1). \tag{1}$$

The following identity is know essentially due to refs.[3]

$$\frac{6}{\pi^2}\sum_{m=1}^{\ell} L\Big(\frac{sin^2\frac{\pi}{\ell+2}}{sin^2\frac{\pi(m+1)}{\ell+2}}\Big) = \frac{3\ell}{\ell+2}, \qquad \text{for } \ell \in \mathbf{Z}_{\ge 1}. \tag{2}$$

In the above, the rhs is the well known value of the central charge for the level ℓ $A_1^{(1)}$ WZW model in conformal field theory. On the other hand, the argument of the dilogarithm has the form $(Q_m)^{-2}$, where Q_m is the $m+1$ dimensional irreducible A_1 character specialized to some "rational point". Thus Eq. 2 is connecting the two important quantities in the A_1-related theory, i.e., the central charge and the specialized character.

In fact, there is a conjectural generalization of the identity Eq. 2 into arbitrary classical simple Lie algebra X_r[4,1].

$$\frac{6}{\pi^2}\sum_{a=1}^{r}\sum_{m=1}^{t_a\ell} L\big(f_m^{(a)}\big) = \frac{\ell dim X_r}{\ell+g}, \qquad \text{for } \ell \in \mathbf{Z}_{\ge 1}. \tag{3}$$

Here, g denotes the dual Coxeter number and t_a is the integer defined as the ratio of the $a-$th Kac and dual Kac label. The argument $0 \le f_m^{(a)} \le 1$ arises through thermodynamic Bethe ansatz (TBA) analysis. Here we shall give its definition only for $X_r = A_r$. (See refs.[1,4,5] for the general case.)

$$f_m^{(a)} = f_m^{(a)}(z=0), \qquad f_m^{(a)}(z) = 1 - \frac{Q_{m+1}^{(a)}(z)Q_{m-1}^{(a)}(z)}{Q_m^{(a)}(z)^2}, \tag{4}$$

where $Q_m^{(a)}(z)$ is the irreducible A_r character with the highest weight $m\Lambda_a$. ($z \in$ dual space of the Cartan subalgebra and Λ_a is the $a-$th fundamental weight. Nodes on the Dynkin diagram are enumerated according to ref.[1].) In general, the quantity $Q_m^{(a)}(z)$ is a Yangian character and we adopt the z dependence as given in ref.[2]. The conjecture (4) was firstly systematically used in the TBA analysis in refs.[6,1].

2. Scaling Dimensions from Dilogarithm

There is a generalization[2] of Eq. 3 so as to include the parafermion scaling dimensions (modulo integer) in CFT $\Delta^{\Lambda}_{\lambda} = \frac{(\Lambda|\Lambda+2\rho)}{2(\ell+g)} - \frac{|\lambda|^2}{2g}$, where Λ is a level ℓ dominant integral weight of $X^{(1)}_r$ and $\lambda \in \Lambda+$ root lattice. This is achieved by considering the specialization $f^{(a)}_m(z = \Lambda)$ instead of the principal one in Eq. 4. Now that the $f^{(a)}_m(\Lambda)$ is a complex number in general, one can consider various analytic continuations $L_{a,m}(x)$ of $L(x)$. Leaving all the technical points[2], we have a conjecture

$$\frac{6}{\pi^2}\sum_{a=1}^{r}\sum_{m=1}^{t_a\ell-1} L_{a,m}\big(f^{(a)}_m(\Lambda)\big) + \text{ Logarithmic terms} = \frac{\ell dim X_r}{\ell+g} - r - 24(\Delta^{\Lambda}_{\lambda} + \text{ integer}). \quad (5)$$

Here, λ depends on the integration contour along which the $L(x)$ is analytically continued. Its explicit form and the logarithmic terms can be found in ref.[2]. See also refs. [7,8,9,10] for some physical aspects. Here we shall only present a conjecture on the value $Q^{(a)}_{t_a\ell}(\Lambda)$, under which the congruence $\lambda \equiv \Lambda$ (modulo root lattice) can be verified directly. Put $k =$ number of the Kac labels a_i $(0 \le i \le r)$ equal to 1. Then we conjecture that $Q^{(a)}_{t_a\ell}(\Lambda)$ is a $k-$th root of unity as

$$\begin{aligned} Q^{(a)}_{t_a\ell}(\Lambda) &= \exp(-2\pi i c(\Lambda)\overline{\gamma}_a/k) \quad \text{for } X_r \ne D_r, \\ &= \exp(2\pi i c_2(\Lambda)\gamma^{(2)}_a/k) \quad \text{for } X_r = D_r,\ r = \text{odd}, \\ &= \exp(2\pi i((c_2(\Lambda) - rc_1(\Lambda))\gamma^{(1)}_a + c_1(\Lambda)\gamma^{(2)}_a)/k) \quad \text{for } X_r = D_r,\ r = \text{even}. \end{aligned} \quad (6)$$

Here, for $\Lambda = \sum_{a=1}^{r}\mu_a\Lambda_a$, we have set $c(\Lambda) = \sum_a \gamma_a\mu_a \mod k$ for $X_r \ne D_r$ and $c_i(\Lambda) = \sum_a \gamma^{(i)}_a\mu_a \mod k/(3-i)$ for $X_r = D_r$, $i = 1,2$ and γ is the rank-dimensional integer vector given by

$$\begin{aligned} &A_r, B_r, C_r, E_6 : \gamma_a = a, \\ &D_r : \gamma^{(1)} = (0,\ldots,0,1,1), \quad \gamma^{(2)} = (2,4,6,\ldots,2(r-2), r-2, r), \\ &E_7 : \gamma = (0,0,0,1,0,1,1), \\ &E_8, F_4, G_2 : \gamma = (0,\ldots,0). \end{aligned} \quad (7)$$

Finally, $\overline{\gamma} = -\gamma$ if $X_r = A_r$ and $\overline{\gamma} = \gamma$ of the dual algebra of X_r if $X_r \ne A_r, D_r$. Eq. 6 is a special solution of $\prod_b Q^{(b)}_{t_b\ell}(\Lambda)^{C_{ab}} = 1$ where C is the Cartan matrix of X_r. From numerical tests it also seems valid that $Q^{(a)}_m(\Lambda) = Q^{(a)}_{t_a\ell}(\Lambda)Q^{(a)}_{t_a\ell-m}(\Lambda)^*$ for $-1 \le m \le t_a\ell + 1$ and $Q^{(a)}_{t_a\ell+1}(\Lambda) = 0$ for any level ℓ dominant integral weight Λ. These are interesting arithmetic properties of the specialized Yangian characters. We note especially that $Q^{(a)}_m(0)$ appears[1] as the high temperature limit of *log*(*entropy*) per site in the TBA system connected to X_r. This implies that $Q^{(a)}_m(0)$ yields the largest eigenvalue of the incidence matrix for a fusion $X^{(1)}_r$ RSOS model.

3. Functional Relations

The Yangian character $Q_m^{(a)}(z)$ is known to satisfy interesting recursion relations[4,5]. For example in $X_r = A_r$ case,

$$Q_m^{(a)}(z)^2 = Q_{m+1}^{(a)}(z)Q_{m-1}^{(a)}(z) + Q_m^{(a+1)}(z)Q_m^{(a-1)}(z) \tag{8}$$

and similar relations are known for all the other algebras. Furthermore there is a "spectral parameter dependent version (or Yang-Baxterization)" of these relations. Below we shall describe it briefly for $X_r = A_r$. In ref.[6] Bazhanov and Reshetikhin wrote down a system of functional relations among the row to row transfer matrices for the fusion $A_r^{(1)}$ model[11].

$$T^{\xi,\eta}(u) = \det\left(T^{\xi,(\eta_i-i+j)\Lambda_1}(u+\eta_1+i-\eta_i-1)\right)_{1\le i,j\le\eta'_1} \tag{9a}$$

$$= \det\left(T^{\xi,\Lambda_{\eta'_i-i+j}}(u+\eta_1-j)\right)_{1\le i,j\le\eta_1}, \tag{9b}$$

where ξ (resp. η) is the Young diagram representing the fusion type in the horizontal (resp. vertical) direction and the $T^{\xi,\eta}(u)$ is transferring the states into the vertical direction. $\eta' = [\eta'_1, \ldots, \eta'_{\eta_1}]$ is the transpose of $\eta = [\eta_1, \ldots, \eta_{\eta'_1}]$ and u is the spectral parameter entering the solution of the Yang-Baxter equation that underlies the model[11]. Eq. 9 is a "quantum analogue" of the 2nd Weyl character formula (Jacobi-Trudi's formula)[6]. (In fact Eq. 9 is different from that in ref.[6] and the alternation is based on a private communication with V.V. Bazhanov. Note that $T^{\xi,\eta}(u)$ may be regarded as representing an eigenvalue thanks to the commutatibity.) Now let η be the a by m rectangular Young diagram corresponding to $m\Lambda_a$ and write the $T^{\xi,\eta}(u)$ as $T_m^{(a)}(u)$. Then from Eq. 9 one can prove the functional relation

$$T_m^{(a)}(u)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u+1) + T_m^{(a+1)}(u)T_m^{(a-1)}(u+1), \tag{10}$$

which is a "Yang-Baxterization" of Eq. 8. From this we find that the combination

$$y_m^{(a)}(u+\frac{a+m}{2}) = \frac{T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u+1)}{T_m^{(a+1)}(u)T_m^{(a-1)}(u+1)} \tag{11}$$

solves essentially the following $U_q(A_r^{(1)})$ functional relation proposed in ref.[2]

$$y_m^{(a)}(u+\frac{1}{2})y_m^{(a)}(u-\frac{1}{2}) = \frac{(1+y_{m+1}^{(a)}(u))(1+y_{m-1}^{(a)}(u))}{(1+y_m^{(a+1)}(u)^{-1})(1+y_m^{(a-1)}(u)^{-1})}. \tag{12}$$

As noted in ref.[2], the above equation is also satisfied by $e^{\epsilon_m^{(a)}(u)}$ ($\epsilon_m^{(a)}(u)$: pseudo energy) in the TBA at ∞ temperature. In this way, one can roughly say that the TBA equation has a solution in terms of "Yang-Baxterized" Yangian characters. It is interesting to note that Eq. 10 can be viewed as a simplest example of the Plücker relation under Eq. 9.

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